Differentiable generalized synchronization of chaos

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We consider simple Lyapunov-exponent-based conditions under which the response of a system to a chaotic drive is a *smooth* function of the drive state. We call this *differentiable generalized synchronization* (DGS). When DGS does not hold, we quantify the degree of nondifferentiability using the Hölder exponent. We also discuss the consequences of DGS and give an illustrative numerical example. [S1063-651X(97)02704-9]

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I. INTRODUCTION

Recently, the concept of *generalized synchronization* has been introduced [1] to characterize the dynamics of a response system that is driven by the output of a chaotic driving system. Generalized synchronization (GS) is said to occur if, ignoring transients, the response $\mathbf{y}$ is uniquely determined by the current drive state $\mathbf{x}$. That is, $\mathbf{y}=\boldsymbol{\phi}(\mathbf{x})$, where $\boldsymbol{\phi}$ is a function of $\mathbf{x}$ for $\mathbf{x}$ on the chaotic attractor of the drive system (the attractor is assumed to be bounded). If GS applies, the $\mathbf{x}$ dynamics can typically be topologically reconstructed from the $\mathbf{y}$ dynamics. (Depending on the dimension of the vector $\mathbf{y}$ and on the fractal dimension of the attractor in $\mathbf{x}$, reconstruction may require formation of a delay coordinate vector from $\mathbf{y}$ [2].)

In the case of continuous time (flows) we can write the combined drive-response system as

$$d\mathbf{x}/dt=\mathbf{f}(\mathbf{x}), \tag{1a}$$

$$d\mathbf{y}/dt=\mathbf{g}(\mathbf{y},\mathbf{h}(\mathbf{x})), \tag{1b}$$

where $\mathbf{x}\in R^k$, $\mathbf{y}\in R^l$, and $\mathbf{f}:R^k\rightarrow R^k$, $\mathbf{h}:R^k\rightarrow R^m$, and $\mathbf{g}:R^l\times R^m\rightarrow R^l$ are continuously differentiable functions. In the case of a discrete time drive-response system, we write

$$\mathbf{x}_{t+1}=\mathbf{F}(\mathbf{x}_t), \tag{1a$'$}$$

$$\mathbf{y}_{t+1}=\mathbf{G}(\mathbf{y}_t,\mathbf{H}(\mathbf{x}_t)), \tag{1b$'$}$$

where we assume $\mathbf{F}$ is invertible, $\mathbf{G}(\mathbf{y},\mathbf{H})$ is invertible in $\mathbf{y}$, and $\mathbf{F}$, $\mathbf{G}$, and $\mathbf{H}$ are continuously differentiable.

Kocarev and Parlitz [3] formulated a condition for the occurrence of GS for the system (1), which, after a slight reformulation, can be stated as follows: GS *occurs if, for all initial* $\mathbf{x}_0$ *in a neighborhood of the chaotic attractor of the drive system, the response system is asymptotically stable*. Recall that the response system is said to be asymptotically stable if there is a region of $\mathbf{y}$ space B such that, for any two initial $\mathbf{y}$ vectors $\mathbf{y}_0^{(1)},\mathbf{y}_0^{(2)}\in B$, we have $\lim_{t\rightarrow\infty}\|\mathbf{y}(t,\mathbf{x}_0,\mathbf{y}_0^{(1)})-\mathbf{y}(t,\mathbf{x}_0,\mathbf{y}_0^{(2)})\|=0$, and $\mathbf{y}(t,\mathbf{x}_0,\mathbf{y}^{(1)})$ and $\mathbf{y}(t,\mathbf{x}_0,\mathbf{y}^{(2)})$ are in the interior of B for t sufficiently large. In the remainder of this paper we assume the response system is asymptotically stable.

There are various situations motivating consideration of GS. The most natural such situation occurs in the one-way synchronization of two oscillators. An important special case of the dynamics (1b) is that of linear coupling,

$$\mathbf{g}(\mathbf{y},\mathbf{h}(\mathbf{x}))=\tilde{\mathbf{f}}(\mathbf{y})+\mathbf{C}\cdot(\mathbf{x}-\mathbf{y}), \tag{2}$$

between two nearly identical oscillatory systems, where $\mathbf{C}$ is a coupling matrix, and $\tilde{\mathbf{f}}$ and $\mathbf{f}$ are close. In cases where the $\mathbf{x}$ dynamics is chaotic and $\mathbf{C}$ is properly chosen, exact stable synchronism, $\mathbf{y}(t)=\mathbf{x}(t)$, occurs *provided* $\mathbf{f}=\tilde{\mathbf{f}}$ (see [1,3,4] and references therein). Experimentally, one cannot expect exact equality of $\mathbf{f}$ and $\tilde{\mathbf{f}}$, and hence one cannot expect exact synchronism. Even so, GS might apply, with essentially the same useful practical consequences as exact synchronism.

Another situation occurs when we cannot observe the system state $\mathbf{x}$ directly, and Eq. (1b) models the response of the measurement apparatus to the system state. Still another example is where the response $\mathbf{y}$ is a linearly filtered version of the input (e.g., see [5]). More generally, we can expand this viewpoint to regard Eq. (1b) as a nonlinear filter.

While knowledge of the existence of a relation of the form $\mathbf{y}=\boldsymbol{\phi}(\mathbf{x})$ is useful, it is often important to also consider the continuity and smoothness of the function $\boldsymbol{\phi}$. For example, it is known in the context of filtering [5] that the relationship between the filtered signal and the system state, although expressible [6] in the form $\mathbf{y}=\boldsymbol{\phi}(\mathbf{x})$, can be such that the attractor reconstructed from $\mathbf{y}$ may have a larger information dimension than the original attractor in $\mathbf{x}$ space. That is, the function $\boldsymbol{\phi}$ may be ''wild'' enough that it changes the attractor's information dimension. If we are interested, for example, in deducing the attractor dimension of the drive system from observations of the response $\mathbf{y}$, this is undesirable. One can also cite other examples where sufficiently smooth $\boldsymbol{\phi}$ is desirable (e.g., obtaining eigenvalues of unstable periodic orbits or Lyapunov exponents of the drive system from observations of the response). Thus we wish to consider a stronger version of GS that we denote *differentiable generalized synchronization* (DGS). By this we simply mean that there is GS and the function $\boldsymbol{\phi}(\mathbf{x})$ is continuously

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differentiable for $\mathbf{x}$ on the chaotic attractor of the drive system. In addition, if DGS does not hold, we wish to quantify the degree of nondifferentiability using the Hölder exponent.

Remark. Note that the attractor of the drive system may be smooth in some directions and fractal in others, and that we are primarily interested in the function $\boldsymbol{\phi}(\mathbf{x})$ evaluated for $\mathbf{x}$ on the attractor of the drive system. In this context we define differentiability of $\boldsymbol{\phi}$ as follows. If there exists a matrix $\nabla_x \boldsymbol{\phi}$, such that for small $\boldsymbol{\delta}$ with $\mathbf{x}+\boldsymbol{\delta}$ on the attractor,

$$\boldsymbol{\phi}(\mathbf{x}+\boldsymbol{\delta}) = \boldsymbol{\phi}(\mathbf{x}) + \boldsymbol{\delta}\cdot\nabla_x\boldsymbol{\phi} + o(\|\delta\|),$$

then we say $\boldsymbol{\phi}$ is differentiable at the point $\mathbf{x}$ on the attractor. Note that this defines differentiability requiring the evaluation of $\boldsymbol{\phi}$ only at points on the drive attractor and that the definition includes directions cutting across the fractal structure of the attractor.

II. HÖLDER EXPONENT AND DGS

Assuming GS applies, we first consider the Hölder exponent $\gamma(\mathbf{x})$ of the function $\boldsymbol{\phi}(\mathbf{x})$ evaluated at the point $(\mathbf{x},\mathbf{y})$, where $\mathbf{x}$ is on the drive attractor and $\mathbf{y}=\boldsymbol{\phi}(\mathbf{x})$. For points $\mathbf{x}$ and $\mathbf{x}+\boldsymbol{\delta}$ on the drive attractor, we define the Hölder exponent $\gamma(\mathbf{x})$ of $\boldsymbol{\phi}(\mathbf{x})$ at $\mathbf{x}$ as

$$\gamma(\mathbf{x}) = \liminf_{\boldsymbol{\delta}\to 0}\{\log\|\boldsymbol{\phi}(\mathbf{x}+\boldsymbol{\delta}) - \boldsymbol{\phi}(\mathbf{x})\|/\log\|\boldsymbol{\delta}\|\}, \tag{3}$$

if the right-hand side is less than one, and $\gamma(\mathbf{x})=1$ if the right-hand side is greater than one.

It can be easily shown that $\gamma(\mathbf{x})>0$ implies that $\boldsymbol{\phi}$ is continuous at $\mathbf{x}$, and that $\boldsymbol{\phi}$ is not differentiable at $\mathbf{x}$ for $\gamma(\mathbf{x})<1$. If $\boldsymbol{\phi}(\mathbf{x})$ is differentiable at $\mathbf{x}$, then $\gamma(\mathbf{x})=1$. If $\boldsymbol{\phi}(\mathbf{x})$ is discontinuous at $\mathbf{x}$, then $\gamma(\mathbf{x})=0$. On the other hand, $\gamma(\mathbf{x})=1$ does not necessarily imply that $\boldsymbol{\phi}(\mathbf{x})$ is differentiable at $\mathbf{x}$, nor does $\gamma(\mathbf{x})=0$ necessarily imply that $\boldsymbol{\phi}(\mathbf{x})$ is discontinuous at $\mathbf{x}$.

We proceed to determine $\gamma(\mathbf{x})$ in terms of the dynamics of the drive and response systems. At each point $\mathbf{x}$ on the drive attractor, and at each corresponding point $\mathbf{y}=\boldsymbol{\phi}(\mathbf{x})$ of the response, we imagine that we evaluate the past-history Lyapunov exponents. That is, for time $T>0$ we look at the T preimage of $\mathbf{x}$, evaluate the finite-time Lyapunov exponents over the orbit segment traveling from the T preimage of $\mathbf{x}$ to $\mathbf{x}$, and then let $T\to\infty$. For almost every point $\mathbf{x}$ with respect to the attractor's natural measure, the *same* set of numbers for the past-history Lyapunov exponents will be found, and these are also the same as the forward Lyapunov exponents at $\mathbf{x}$. (We call such points *typical*.) However, usually there is also a dense set of points $\mathbf{x}$ on the attractor for which the past-history Lyapunov exponents are different from those of typical points. The set of all these atypical points has zero natural measure, but these points are nevertheless significant in our considerations. Thus, in general, we must regard the past-history Lyapunov exponents as $\mathbf{x}$ dependent. The simplest example illustrating $\mathbf{x}$ dependence of the past-history Lyapunov exponents is the case where $\mathbf{x}$ lies precisely on the unstable manifold of an unstable periodic orbit in the chaotic attractor of the drive system. In that case, as $T\to\infty$, the T preimage of $\mathbf{x}$ approaches the periodic orbit. Thus the past-history exponents will be those of the periodic orbit, which in general are different from those of typical orbits on the attractor.

To begin our quantitative discussion, we first consider the case of a map where the drive system attractor has a set of past-history Lyapunov exponents that at each point $\mathbf{x}$ on the attractor consists of $k-1$ positive exponents and one negative exponent, denoted by $-h_d(\mathbf{x})$. In this case we expect the attractor structure at each point $\mathbf{x}$ to be smooth in the $k-1$ expanding directions and to be fractal in the contracting direction corresponding to the Lyapunov exponent $-h_d(\mathbf{x})$. This situation applies for many examples encountered in practice (e.g., for chaotic attractors of invertible two-dimensional maps such as the Hénon map, the Ikeda map, etc.); the considerations also translate readily to flows with one negative Lyapunov exponent. Let $-h_r(\mathbf{x})$ denote the least-negative response-system past-history Lyapunov exponent corresponding to the point $(\mathbf{x},\mathbf{y}=\boldsymbol{\phi}(\mathbf{x}))$ (recall that we assume the response to be asymptotically stable, implying that all response exponents are nonpositive; we consider the case where the Lyapunov exponents of the response system are negative).

Our principal results for the case where the drive has only one negative exponent are the following. Their application to the general case, in which there may be more than one negative exponent, is discussed later.

(i) The Hölder exponent of the function $\boldsymbol{\phi}(\mathbf{x})$ at a point $\mathbf{x}$ on the drive attractor is one if $h_r(\mathbf{x})\geqslant h_d(\mathbf{x})$. For typical systems, if $h_r(\mathbf{x})<h_d(\mathbf{x})$ the Hölder exponent is

$$\gamma(\mathbf{x}) = h_r(\mathbf{x})/h_d(\mathbf{x}). \tag{4}$$

We discuss the meaning of the phrase ''typical system'' subsequently. [Including the atypical cases, we have that, in general, if $h_r(\mathbf{x})<h_d(\mathbf{x})$, then $\gamma(\mathbf{x})\geqslant h_r(\mathbf{x})/h_d(\mathbf{x})$.]

(ii) The function $\boldsymbol{\phi}(\mathbf{x})$ is differentiable for all $\mathbf{x}$ on the drive attractor (i.e., DGS applies), if

$$h_r(\mathbf{x}) > h_d(\mathbf{x}) \tag{5}$$

for all $\mathbf{x}$ on the drive attractor.

Recall that the existence of a Hölder exponent, $\gamma(\mathbf{x})>0$, claimed in (i), means that $\boldsymbol{\phi}$ is continuous for points $\mathbf{x}$ on the drive-system attractor. In some cases, results in the literature [7] on the existence of invariant manifolds and their persistence under perturbation yield conclusions similar to (but generally weaker than) (i) and (ii).

We now give a heuristic argument for (i) and (ii). For simplicity, consider the case where $\mathbf{x}$ is a point on a period p unstable periodic orbit embedded in the drive attractor. Let $\boldsymbol{\delta}_0$ denote an initial displacement from the point $\mathbf{x}$, where $\mathbf{x}+\boldsymbol{\delta}_0$ is on the drive attractor. Using Eq. (1a) or Eq. (1a′) take the displaced point forward in time by the amount n, $\boldsymbol{\delta}_0\to\boldsymbol{\delta}_n$, where $n=mp$ and m is an integer. Similarly, take the point $\mathbf{y}+\delta\mathbf{y}_0=\boldsymbol{\phi}(\mathbf{x}+\boldsymbol{\delta}_0)$ forward the same time n using Eq. (1b) or Eq. (1b′), $\delta\mathbf{y}_0\to\delta\mathbf{y}_n$. According to the Hartman-Grobman theorem, there exists a change of variables that makes the dynamics linear in a finite region about the periodic orbit (assuming generic eigenvalues). Thus it suffices to consider the dynamics as linear. Assume $\boldsymbol{\delta}_0$ to be chosen to lie in the eigendirection of the linearized drive system corresponding to the eigenvalue yielding the contractive

$\mathbf{x}$-Lyapunov exponent, $-h_d(\mathbf{x})$. We wish to examine the behavior of $\delta\mathbf{y}_0=\boldsymbol{\phi}(\mathbf{x}+\boldsymbol{\delta}_0)-\boldsymbol{\phi}(\mathbf{x})$ as it is iterated forward in time. To do this we write the linearized drive-response system as $\mathbf{z}_{n+1}=\mathbf{Z}\mathbf{z}_n$ where

$$\mathbf{z}=\begin{bmatrix}\mathbf{x}\\ \mathbf{y}\end{bmatrix},$$

$$\mathbf{Z}=\begin{bmatrix}\mathbf{X} & \mathbf{O}\\ \mathbf{W} & \mathbf{Y}\end{bmatrix}. \tag{6}$$

Here $\mathbf{x}$ is a k vector, $\mathbf{y}$ is an l vector, $\mathbf{z}$ is a $(k+l)$ vector, $\mathbf{O}$ is the $l\times k$ zero matrix, and the matrices $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{W}$ have dimensions $k\times k$, $l\times l$, and $k\times l$, respectively. Generically, $\mathbf{X}$ and $\mathbf{Y}$ have distinct eigenvalues,

$$\mathbf{X}\mathbf{e}_{xp}=\lambda_{xp}\mathbf{e}_{xp}, \quad p=1,2,\ldots,k, \tag{7}$$

$$\mathbf{Y}\mathbf{e}_{yq}=\lambda_{yq}\mathbf{e}_{yq}, \quad q=1,2,\ldots,l. \tag{8}$$

The matrix $\mathbf{Z}$ has eigenvalues λ_{xp}, λ_{yq} with corresponding $(k+l)$-dimensional eigenvectors $\mathbf{e}_{zxp}$ and $\mathbf{e}_{zyq}$,

$$\mathbf{Z}\mathbf{e}_{zxp}=\lambda_{xp}\mathbf{e}_{zxp}, \quad p=1,\ldots,k, \tag{9}$$

$$\mathbf{Z}\mathbf{e}_{zyq}=\lambda_{yq}\mathbf{e}_{zyq}, \quad q=1,\ldots,l. \tag{10}$$

The eigenvectors are

$$\mathbf{e}_{zxp}=\begin{bmatrix}\mathbf{e}_{xp}\\ \hat{\mathbf{e}}_p\end{bmatrix},$$

$$\mathbf{e}_{zyq}=\begin{bmatrix}\mathbf{O}\\ \mathbf{e}_{yq}\end{bmatrix}, \tag{11}$$

where $\hat{\mathbf{e}}_p=(\lambda_{xp}\mathbf{1}_l-\mathbf{Y})^{-1}\mathbf{W}\mathbf{e}_{xp}$, $\mathbf{O}$ is the k-dimensional zero vector, the $l\times l$ unit matrix is denoted $\mathbf{1}_l$ and the indicated inverse exists assuming the generic condition $\lambda_{xp}\neq\lambda_{yq}$ for all p and q. Our original question of what happens to $\delta\mathbf{y}_0$ as it is iterated can be addressed by considering the entire system with initial displacement

$$\delta\mathbf{z}_0=\begin{bmatrix}\boldsymbol{\delta}_0\\ \delta\mathbf{y}_0\end{bmatrix},$$

where the initial $\mathbf{x}$ displacement is $\boldsymbol{\delta}_0=\delta_0\mathbf{e}_{xd}$ and $\mathbf{e}_{xd}$ is the $\mathbf{x}$ eigenvector corresponding to the contractive eigendirection with eigenvalue λ_{xd}, $|\lambda_{xd}|=\exp(-h_d)$. Writing $\delta\mathbf{z}_0$ as

$$\delta\mathbf{z}_0=\delta_0\begin{bmatrix}\mathbf{e}_{xd}\\ \hat{\mathbf{e}}_d\end{bmatrix}+\begin{bmatrix}\mathbf{O}\\ \delta\mathbf{y}_0-\delta_0\hat{\mathbf{e}}_d\end{bmatrix}, \tag{12}$$

we see that the first vector on the right-hand side of Eq. (12) evolves with time as $|\lambda_{xd}|^n=\exp(-nh_d)$. Typically we expect that $\delta\mathbf{y}_0-\delta_0\hat{\mathbf{e}}_d$ has no special relationship to the eigendirections of $\mathbf{Y}$. In this case all of the eigendirections of $\mathbf{Y}$ will be ''excited.'' For large time n the response eigenvalue of largest magnitude will dominate, $|\lambda_{yr}|^n=\exp(-nh_r)$. Thus, for large n, $\delta\mathbf{y}_n$ consists of two components: one excited by the first vector on the right-hand side of Eq. (12) decaying as $\exp(-nh_d)$, and one excited by the second vector on the right-hand side of Eq. (12) decaying as $\exp(-nh_r)$. Which of these two behaviors dominates at large n is determined by whether $h_d>h_r$ or $h_d<h_r$. We have after a large number of iterates

$$\delta\mathbf{y}_0\to\delta\mathbf{y}_n\sim\begin{cases}\exp[-h_r(\mathbf{x})n], & \text{if } h_r(\mathbf{x})<h_d(\mathbf{x})\\ \exp[-h_d(\mathbf{x})n], & \text{if } h_r(\mathbf{x})\geqslant h_d(\mathbf{x})\end{cases}, \tag{13a}$$

$$\boldsymbol{\delta}_0\to\boldsymbol{\delta}_n\sim\exp[-h_d(\mathbf{x})n], \tag{13b}$$

where $\delta\mathbf{y}_n=\boldsymbol{\phi}(\mathbf{x}+\boldsymbol{\delta}_n)-\boldsymbol{\phi}(\mathbf{x})$. For $h_r(\mathbf{x})<h_d(\mathbf{x})$, we have from Eq. (13),

$$\|\boldsymbol{\phi}(\mathbf{x}+\boldsymbol{\delta}_n)-\boldsymbol{\phi}(\mathbf{x})\|\sim\|\boldsymbol{\delta}_n\|^{\gamma(\mathbf{x})}, \tag{14}$$

where $\gamma(\mathbf{x})$ is given by Eq. (4). Thus, since $\boldsymbol{\delta}_n\to 0$ as $n\to\infty$, Eq. (14) yields Eq. (4). Similarly, for the case $h_r(\mathbf{x})\geqslant h_d(\mathbf{x})$, Eqs. (13) again yield Eq. (14), but with $\gamma(\mathbf{x})=1$. This argument applies for $\mathbf{x}$ being a periodic point. Another argument, not given here, shows that the same result applies to the case where $\mathbf{x}$ is any nonperiodic point on the attractor. (In the case of a periodic orbit the past-history and forward Lyapunov exponents are the same. When $\mathbf{x}$ is nonperiodic they are not necessarily the same, and it is the past-history exponents that are relevant.)

In order to demonstrate (ii), we note that when $h_r(\mathbf{x})>h_d(\mathbf{x})$, taking $\boldsymbol{\delta}_0=\delta_0\mathbf{e}_{xp}$, for any $p=1,2,\ldots,k$, the linear dynamics (7) always results in

$$\delta\mathbf{z}_n\cong\lambda_{xp}^n\delta_0\begin{bmatrix}\mathbf{e}_{xp}\\ \hat{\mathbf{e}}_p\end{bmatrix}, \tag{15}$$

for sufficiently large n. Thus, since the $\mathbf{e}_{xp}$ generically span the x space, we have that the derivatives of $\boldsymbol{\phi}(\mathbf{x})$ exist and are given by

$$\mathbf{e}_{xp}\cdot\boldsymbol{\nabla}_x\boldsymbol{\phi}=\hat{\mathbf{e}}_p, \quad p=1,2,\ldots,k. \tag{16}$$

[In the case of complex λ_{xp}, we can take the real and imaginary parts of (16).]

In the above discussion obtaining (i) it was assumed that the drive-response system was typical in that the chosen direction for $\boldsymbol{\delta}_0$ (namely, $\boldsymbol{\delta}_0=\delta_0\mathbf{e}_{xd}$) yields $\delta\mathbf{y}_0$, which results in a nonzero component of $\delta\mathbf{y}_0-\delta_0\hat{\mathbf{e}}_d$ along the response-system eigendirection corresponding to the exponent $-h_r(\mathbf{x})$. As an example where this is not the case, consider the situation where the response system is of the form (2) and the drive and response are exactly matched, $\tilde{\mathbf{f}}=\mathbf{f}$. In this case we can have exact synchronism, $\mathbf{y}=\mathbf{x}$ [i.e., $\boldsymbol{\phi}(\mathbf{x})=\mathbf{x}$]. Thus, even though Eq. (5) may be violated, the surface $\mathbf{y}=\boldsymbol{\phi}(\mathbf{x})$ is still smooth (it is the hyperplane $\mathbf{x}=\mathbf{y}$). However, a generic perturbation of the function $\tilde{\mathbf{f}}$ away from $\mathbf{f}$ restores the validity of Eq. (4). On the other hand, we note that, if this perturbation is small, the resulting component of $\delta\mathbf{y}_0$ along the response-system eigendirection corresponding to the exponent $-h_r(\mathbf{x})$ is expected to be small (it is zero when the perturbation is zero). Thus in this case we might, for example, expect a $\boldsymbol{\delta}$ dependence of the form $\|\boldsymbol{\phi}(\mathbf{x}+\boldsymbol{\delta})-\boldsymbol{\phi}(\mathbf{x})\|\cong K\|\boldsymbol{\delta}\|^{\gamma(\mathbf{x})}$ to apply with a relatively small value of K. A small enough value of K would have the effect that, in the presence of limited precision measurement and/or small noise, the consequences of nondifferentiability may be unobservable.

Note that having only one negative drive exponent implies fractal structure of the drive attractor in the eigendirection corresponding to $-h_d(\mathbf{x})$. Thus our stipulation that we can pick a small $\boldsymbol{\delta}_0$ aligned along the contracting direction such that $\mathbf{x}+\boldsymbol{\delta}_0$ is a point on the attractor can be satisfied. In the case where there are several contracting directions for the response system, the attractor may be "empty" in some of the contracting directions (i.e., for some of the contracting directions, there may be no small $\boldsymbol{\delta}_0$ aligned along that contracting direction such that both $\mathbf{x}$ and $\mathbf{x}+\boldsymbol{\delta}_0$ are on the attractor). In that case, by slightly extending our considerations, our previous discussion can still be applied. To do this, at each point $\mathbf{x}$ on the attractor we consider those negative drive exponents that correspond to eigendirections that locally intersect the attractor at more than a single point ("nonempty" eigendirections), and we take the most negative of those, which we now denote by $-h_d(\mathbf{x})$. In terms of this new designation of $h_d(\mathbf{x})$, our statements (i) and (ii) are still expected to apply. The reason for taking the most negative intersecting drive exponent is that the definition of $\gamma(\mathbf{x})$, Eq. (3), specifies a limit inferior over $\boldsymbol{\delta}$, and $\gamma(\mathbf{x})$ from Eq. (4) is smallest for larger $h_d(\mathbf{x})$. To see that there may be contracting directions that are empty in the sense that they intersect the attractor only at a single point, recall that high-dimensional systems can often be shown to possess lower-dimensional "inertial manifolds" [8] such that there is a dynamical system for state points in the inertial manifold, and this dynamical system yields all the ergodic invariant sets of the original higher-dimensional system (in particular, its chaotic attractors). In that case contracting eigendirections transverse to the inertial manifold are clearly empty. In the absence of knowledge as to which of the contracting-drive eigenvalues are empty, statements (i) and (ii) are still useful in that use of the most contracting-drive Lyapunov exponent in place of $h_d(\mathbf{x})$ provides a lower bound on $\gamma(\mathbf{x})$ [statement (i)], and a *sufficient* condition for DGS [Eq. (5)].

Finally, we remark that the condition for DGS can sometimes be verified in terms of finite-time Lyapunov exponents. The time-T Lyapunov exponents of a system are defined to be $1/T$ times the logarithms of the singular values of the Jacobian matrix of the time-T map of the system. (The singular values of a matrix $\mathbf{M}$ are the square roots of the eigenvalues of $\mathbf{M}\mathbf{M}^T$, where $\mathbf{M}^T$ is the transpose of $\mathbf{M}$.) If, for some T and for all $\mathbf{x}$ on the attractor, the most negative time-T Lyapunov exponent of the drive system at $\mathbf{x}$ exceeds the least-negative time-T Lyapunov exponent of the response system at $[\mathbf{x},\boldsymbol{\phi}(\mathbf{x})]$, then we can show that condition (5), and hence DGS, holds. (It may sometimes be possible to verify this finite-time condition for a single iteration in the case of a map or an infinitesimal time step in the case of a flow.)

III. AN EXAMPLE

We now consider a simple example. The drive system is given by a generalized baker's map [9], which takes the unit square, $0\leqslant x^{(1)}<1$, $0\leqslant x^{(2)}<1$, to itself,

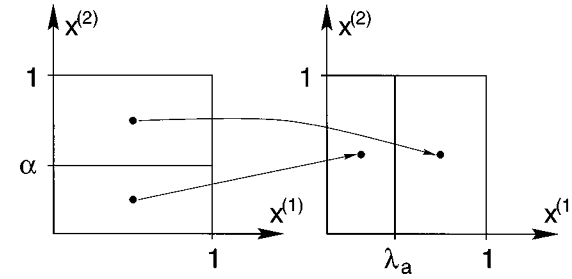

FIG. 1. The generalized baker's map, Eqs. (17).

$$x_{n+1}^{(1)}=\begin{cases}\lambda_a x_n^{(1)}, & \text{if } x_n^{(2)}<\alpha\\ \lambda_1+\lambda_b x_n^{(1)}, & \text{if } x_n^{(2)}\geqslant\alpha'\end{cases} \tag{17a}$$

$$x_{n+1}^{(2)}=\begin{cases}\dfrac{1}{\alpha}\,x_n^{(2)}, & \text{if } x_n^{(2)}<\alpha\\ \dfrac{1}{\beta}\,(x_n^{(2)}-\alpha), & \text{if } x_n^{(2)}<\alpha\end{cases} \tag{17b}$$

where $\alpha+\beta=1$ and we also take $\lambda_a+\lambda_b=1$. See Fig. 1. The response system assumed to be of the form

$$y_{n+1}=\lambda y_n+x_n^{(1)}, \tag{18}$$

can be considered as a discrete version of a low-pass filter [10]. The attractor for Eq. (17) has a natural measure that is uniform in $x^{(2)}$ and varies wildly in the $x^{(1)}$ direction provided that $\lambda_a\neq\alpha$. The box-counting dimension of the attractor is $D_0=2$, since typical trajectories are dense in the unit square by virtue of $\lambda_a+\lambda_b=1$. The past-history Lyapunov exponents for the drive system evaluated at a point $\mathbf{x}=(x^{(1)},x^{(2)})$ are [9]

$$h_1(\mathbf{x})=a(\mathbf{x})\ln\frac{1}{\alpha(\mathbf{x})}+b(\mathbf{x})\ln\frac{1}{\beta(\mathbf{x})}>0 \tag{19a}$$

$$h_2(\mathbf{x})=a(\mathbf{x})\ln\lambda_a+b(\mathbf{x})\ln\lambda_b<0, \tag{19b}$$

where

$$a(\mathbf{x})=\lim_{n\to\infty}\frac{n_a(\mathbf{x})}{n}, \tag{20a}$$

$$b(\mathbf{x})=\lim_{n\to\infty}\frac{n_b(\mathbf{x})}{n}. \tag{20b}$$

In Eq. (20), $n_a(\mathbf{x})$ $[n_b(\mathbf{x})]$ is the number of times the first n preiterates from $\mathbf{x}$ are in $x^{(2)}<\alpha$ $[x^{(2)}\geqslant\alpha]$. [Note that $a(\mathbf{x})+b(\mathbf{x})=1$.] By virtue of the symbolic dynamics for Eq. (17), there are orbits that visit the regions $x^{(2)}<\alpha$ and $x^{(2)}\geqslant\alpha$ in any order. Thus any value for $a(\mathbf{x})$ in [0,1] can be attained by proper choice of $\mathbf{x}$. On the other hand, if $\mathbf{x}$ is randomly chosen in the area of the unit square $(0\leqslant x^{(1,2)}\leqslant1)$, we have $a(\mathbf{x})=\alpha$ and $b(\mathbf{x})=\beta$ with probability one [i.e., the natural measure of the region $x^{(2)}<\alpha$ $(x^{(2)}\geqslant\alpha)$ is α (β)]. Thus the Lyapunov exponents for typical points on the drive attractor are

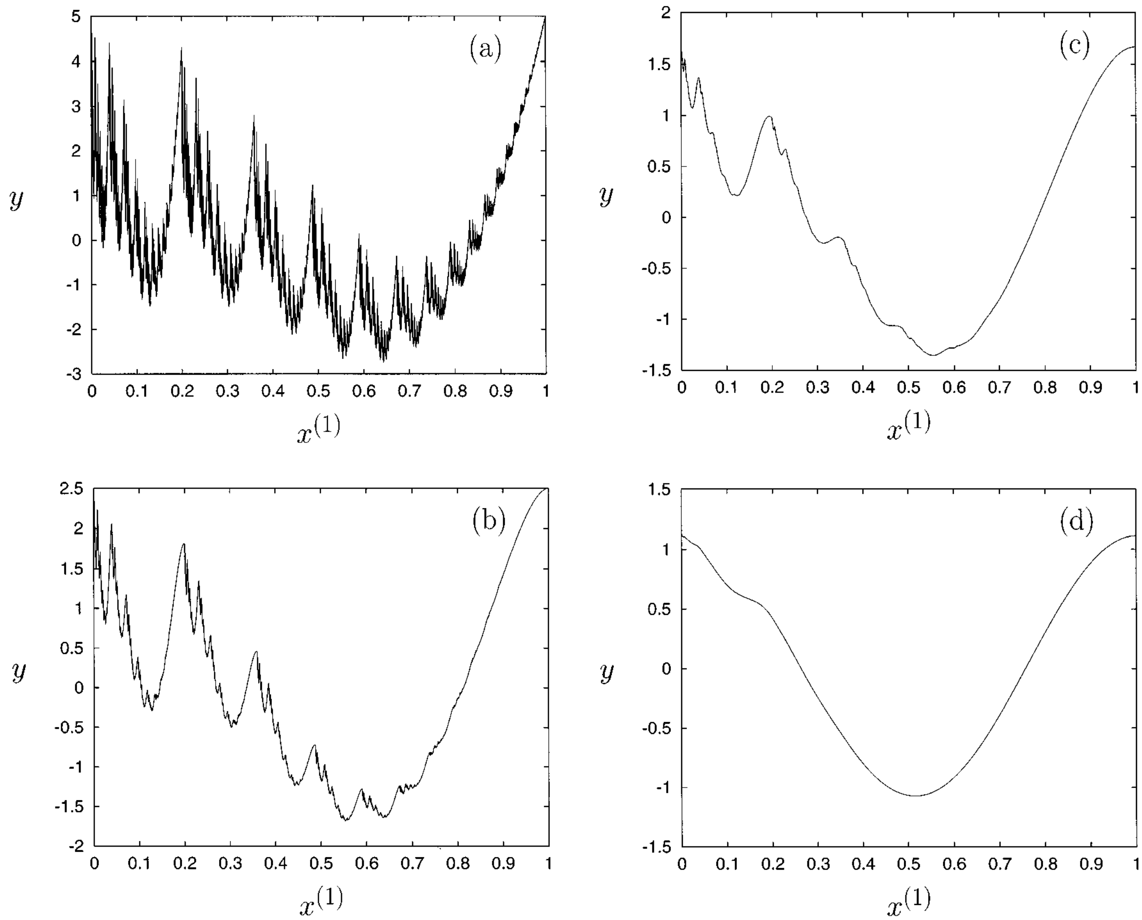

FIG. 2. y vs $x^{(1)}$ for Eqs. (17) and (18).

$$\overline{h_1}=\alpha\ \ln\frac{1}{\alpha}+\beta\ \ln\frac{1}{\beta}>0, \tag{21a}$$

$$\overline{h_2}=\alpha\ \ln\lambda_a+\beta\ \ln\lambda_b<0. \tag{21b}$$

Hence, although the box-counting dimension of the drive attractor is $D_0=2$, by the Kaplan-Yorke formula [9,11], its information dimension is between 1 and 2,

$$D_1=1+(\overline{h_1}/|\overline{h_2}|). \tag{22}$$

Application of the Kaplan-Yorke formula to the combined drive-response system Eqs. (17) and (18) again yields Eq. (22) provided that $h_r>|\overline{h_2}|$ (where $h_r=\ln\lambda^{-1}$), but yields a larger value of the dimension if $h_r<|\overline{h_2}|$. Figure 2(a) shows a numerical computation of the surface $y=\phi(\mathbf{x})$ for a case satisfying $h_r=\ln\lambda^{-1}<|\overline{h_2}|$ [$\lambda_a=0.2$, $\lambda=0.8$]. The resulting curve is fractal as indicated by its very wrinkled appearance. (Note that since ϕ is independent of $x^{(2)}$ and α, it suffices to plot y versus $x^{(1)}$ and Fig. 2(a) is valid for all $0\leq\alpha<1$.)

Now consider the case where $h_r=\ln\lambda^{-1}>|\overline{h_2}|$. In this case the filter does not change the Kaplan-Yorke dimension; i.e., the attractor of the combined drive-response system, Eqs. (17) and (18), still has the information dimension given by Eq. (22). On the other hand, even though $h_r>|\overline{h_2}|$, there is still the possibility that there are points $\mathbf{x}$ on the drive attractor at which $h_r<|h_2(\mathbf{x})|$. In particular, if $\lambda_a<\lambda_b$, then by Eq. (19b), we see that, depending on $\mathbf{x}$, $|h_2(\mathbf{x})|$ can attain any value in the range

$$\ln(1/\lambda_a)\geqslant|h_2(\mathbf{x})|\geqslant\ln(1/\lambda_b). \tag{23}$$

We now consider the case where

$$\ln(1/\lambda_a)>h_r>|\overline{h_2}|. \tag{24}$$

In this case there are points $\mathbf{x}$ at which the Hölder exponent predicted by Eq. (4) is less than one. The natural measure of these points is zero, but they are dense in the attractor. Thus, although the information dimension is preserved, the surface $y=\phi(\mathbf{x})$ is still nonsmooth. Figures 2(b) and 2(c) show the results of numerical computations of the surface $y=\phi(\mathbf{x})$ for two cases satisfying Eq. (24) [$\lambda_a=0.2$, $\lambda=0.6$ for Fig. 2(b), $\lambda_a=0.2$, $\lambda=0.4$ for Fig. 2(c), and we assume $\alpha\leqslant0.2$ in both cases [12]]. The effect of the dense set where $\gamma(\mathbf{x})<1$ clearly manifests itself in the plot shown in Fig. 2(b) giving the surface an extremely wrinkled appearance. The other case satisfying Eq. (24), Fig. 2(c) appears less wrinkled since h_r is closer to $\ln(1/\lambda_a)$. Finally Fig. 2(d) shows a case where $h_r>\ln(1/\lambda_a)$ ($\lambda_a=0.2$, $\lambda=0.1$) in which case the function $y=\phi(\mathbf{x})$ is predicted to be differentiable everywhere. As predicted, the curve in Fig. 2(d) appears to be smooth.

Note that, as we increase λ from zero, $\phi(\mathbf{x})$ first loses differentiability as λ passes through λ_a, which is the Lyapunov number of the period one unstable periodic orbit $(x^{(1)},x^{(2)})=(0,0)$. More generally, consider the system (1a′),(1b′), where the k-dimensional drive system is uniformly expanding in k-1 directions and is uniformly contracting in one direction, and the response system is a linear filter of the form $\mathbf{y}_{n+1}=\mathbf{\Lambda}\mathbf{y}_n+\mathbf{A}\mathbf{x}_n$, where $\mathbf{\Lambda}$ and $\mathbf{A}$ are matrices. Let $\lambda<1$ denote the magnitude of the largest eigenvalue of $\mathbf{\Lambda}$. Now say that the parameters of the filter are varied. We conjecture that the bifurcation at which $\boldsymbol{\phi}(\mathbf{x})$ first becomes nondifferentiable typically occurs as λ increases through a critical value that is determined by a special low-period periodic orbit which has the smallest ''contracting'' Lyapunov number among all the periodic orbits on the attractor [13], and the critical λ is just this minimum Lyapunov number.

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[6] In the case of a discrete-time, causal, low-pass filter $y_{n+1}=\lambda(y_n+\mathbf{k}\cdot\mathbf{x}_n)$, where $|\lambda|<1$ and $\mathbf{k}$ is a constant vector, we can write y_n as $y_n=\Sigma_{m=1}^{\infty}\lambda^m\mathbf{k}\cdot\mathbf{x}_{n-m}$. Since $\mathbf{x}_n$ evolves from an invertible dynamical system, $\mathbf{x}_{n+1}=\mathbf{F}(\mathbf{x}_n)$, we have $\phi(\mathbf{x})=\Sigma_{m=1}^{\infty}\lambda^m\mathbf{k}\cdot\mathbf{F}^{-m}(\mathbf{x})$. This sum converges since $|\lambda|<1$, and it is a continuous function of $\mathbf{x}$ if $\mathbf{F}(\mathbf{x})$ is continuous.

[7] R. J. Sacker [J. Math. Mech. **18**, 705 (1969)] and N. Fenichel [Indiana Univ. Math. J. **21**, 193 (1971)] obtain Lyapunov exponent-based conditions analogous to (ii) for the persistence of smooth invariant manifolds under perturbation; these results imply (ii) in the case where the drive attractor is a compact manifold and the response system is linear in $\mathbf{y}$. M. W. Hirsch and C. C. Pugh [Proc. Symp. Pure Math. **14**, 133 (1970)] and M. W. Hirsch, C. C. Pugh, and M. Shub [*Invariant Manifolds*, Lecture Notes in Mathematics Vol. 583 (Springer-Verlag, New York, 1977)] give results like the lower bound on $\gamma(\mathbf{x})$ in (i) and (ii), but requiring uniform (Lipschitz-like) bounds on the contraction rates, again assuming in the case of (ii) that the drive attractor is a compact manifold. More recently J. Stark has considered the case of general drive attractors and has independently obtained results like (i) and (ii) concerning the regularity of $\boldsymbol{\phi}$ at ''almost every'' point; we, on the other hand, consider the regularity at all points.

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[13] That the attainment of the minimum-contracting Lyapunov number typically occurs on a low-period periodic orbit is implied by the work in B. R. Hunt and E. Ott, Phys. Rev. Lett. **76**, 2254 (1996).